SECTION 14.2: THE CALCULUS OF VECTOR-VALUED FUNCTIONS

RECALL: Given a function f, the derivative of f is: $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$, provided the limit exists.

We can 'vectorize' the above definition to get:

DEFINITION: Given a v.v.f. $\vec{r}$, the derivative of r is: $\vec{r}'(t) = \lim\limits_{h\to 0} \dfrac{\vec{r}(t+h)-\vec{r}(t)}{h}$, provided the limit exists.

NOTE: Since subtraction, scalar multiplication, and limits go component-wise, so do derivatives of v.v.f.'s.

THEOREM: If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ where f, g, and h are differentiable, then $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t)\rangle$.

EXAMPLE 1: Let $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), \cos(2t)\rangle$.

1. Find and simplify $\vec{r}\,'(t)$.

Ans: $\vec{r}\,'(t) = \langle -2\sin(t), 2\cos(t), -2\sin(2t)\rangle$

2. Find and simplify $\vec{r}(\pi)$ and $\vec{r}\,'(\pi)$.

Ans: $\vec{r}(\pi) = \langle -2, 0, 1\rangle$ and $\vec{r}\,'(\pi) = \langle 0, -2, 0\rangle$

3. Find the vector form of the line containing the point corresponding to $t = \pi$ with direction $\vec{r}\,'(\pi)$.

Ans: $\vec{L}(t) = \langle -2, -2t, 1\rangle$

4. Graph $\vec{r}(t)$ and the line you found together. What do you notice?

DEFINITION: If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ is differentiable at $t = a$ then the **tangent line** at $t = a$ is given by:

$$\vec{L}(t) = \vec{r}(a) + \vec{r}\,'(a)\,t = \big\langle f(a) + f'(a)\,t, g(a) + g'(a)\,t, h(a) + h'(a)\,t\big\rangle$$

PROPERTIES OF THE DERIVATIVE: Suppose $\vec{r}$, $\vec{u}$, and f are differentiable functions and k a scalar.

- **SUM / DIFFERENCE** $D_t\left[\vec{u}(t) \pm \vec{r}(t)\right] = \vec{u}\,'(t) \pm \vec{r}\,'(t)$
- **SCALAR MULTIPLE:** $D_t\left[k\,\vec{r}(t)\right] = k\,\vec{r}\,'(t)$
- **PRODUCT RULE, SCALAR FUNCTION:** $D_t\left[f(t)\,\vec{r}(t)\right] = f'(t)\,\vec{r}(t) + f(t)\,\vec{r}\,'(t)$
- **PRODUCT RULE, DOT PRODUCT:** $D_t\left[\vec{u}(t) \cdot \vec{r}(t)\right] = \vec{u}\,'(t) \cdot \vec{r}(t) + \vec{u}(t) \cdot \vec{r}\,'(t)$
- **PRODUCT RULE, CROSS PRODUCT:** $D_t\left[\vec{u}(t) \times \vec{r}(t)\right] = \vec{u}\,'(t) \times \vec{r}(t) + \vec{u}(t) \times \vec{r}\,'(t)$
- **QUOTIENT RULE, SCALAR FUNCTION:** $D_t\left[\dfrac{\vec{r}(t)}{f(t)}\right] = \dfrac{f(t)\,\vec{r}\,'(t) - f'(t)\,\vec{r}(t)}{[f(t)]^2}$
- **CHAIN RULE:** $D_t\left[\vec{r}(f(t))\right] = \vec{r}\,'(f(t))\,f'(t)$

THEOREM: If $\|\vec{r}(t)\|$ is constant, then $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$.

PROOF:

EXAMPLE 2: Let $\hat{T}(t) = \dfrac{\langle 2t, 1\rangle}{\sqrt{4t^2+1}}$.

1. Verify $\|\hat{T}(t)\| = 1$.

2. Find and simplify $\hat{T}\,'(t)$.

Ans: $\hat{T}\,'(t) = \left\langle \dfrac{2}{(4t^2+1)^{3/2}}, -\dfrac{4t}{(4t^2+1)^{3/2}} \right\rangle$

3. Verify: $\hat{T}(t) \cdot \hat{T}\,'(t) = 0$.

DEFINITIONS:

- A v.v.f. $\vec{r}$ is called **smooth** of $\vec{r}\,'$ is continuous and $\|\vec{r}\,'(t)\| \neq 0$.
- Given a smooth v.v.f. $\vec{r}$, the **unit tangent vector** $\hat{T}$ is defined as: $\hat{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$.

NOTE: $\hat{T}(t)$ is the direction of the curve traced out by $\vec{r}(t)$.

EXAMPLE 3: Let $\vec{r}(t) = \langle 5\cos(t), 5\sin(t) \rangle$.

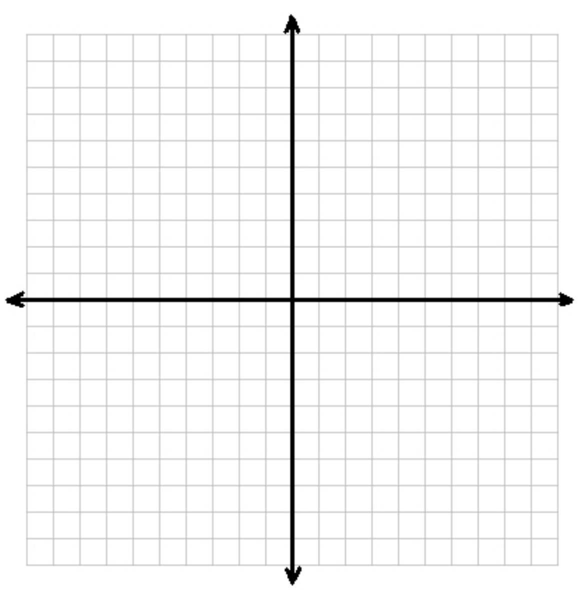
1. Find and simplify an expression for $\hat{T}(t)$.

Ans: $\hat{T}(t) = \langle -\sin(t), \cos(t) \rangle$

2. Find $\hat{T}(0)$, $\hat{T}\left(\frac{\pi}{2}\right)$, $\hat{T}(\pi)$, and $\hat{T}\left(\frac{3\pi}{2}\right)$.

Ans: $\hat{T}(0) =< 0, 1 >$, $\hat{T}\left(\frac{\pi}{2}\right) =< -1, 0 >$, $\hat{T}(\pi) =< 0, -1 >$, and $\hat{T}\left(\frac{3\pi}{2}\right) =< 1, 0 >$

3. Graph $\vec{r}(t)$ on the axes below along along with the values of $\hat{T}(t)$ you found above.
   Plot $\hat{T}(t)$ so that the initial point of $\hat{T}(t)$ the terminal point of the corresponding vector $\vec{r}(t)$.
   For example, plot $\hat{T}(0)$ starting at the position on the curve corresponding to $\vec{r}(0)$, etc.



INTEGRATION OF VECTOR-VALUED FUNCTIONS:

Since differentiation of v.v.f.s goes component-wise, so does integration.

EXAMPLE 4:

1. Find the component form of: $\displaystyle\int_0^{\pi} \left[3\cos(t)\,\hat{\jmath} - 3\sin(t)\,\hat{\imath} + 2t\,\hat{k}\right]\, dt$

Ans: $\left\langle -6, 0, \pi^2 \right\rangle$

2. Find a v.v.f. $\vec{r}(t)$ if $\vec{r}\,'(t) = \left\langle 2t, \dfrac{1}{t} \right\rangle$ and $\vec{r}(1) = \langle 2, 3 \rangle$

Ans: $\vec{r}(t) = \left\langle t^2 + 1, \ln(t) + 3 \right\rangle$

HOMEWORK: Section 14.2: 9 - 81 every other odd.